

From Unate to Uniform Boolean Networks via Switch Equivalence

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Abstract. Unate Boolean networks (BN) can be represented by signed dependency graphs, where the sign of each arc encodes whether one variable has an increasing or decreasing influence on the update of another. Uniform BN are a subclass of unate BN which includes well-studied families of BN, and which is characterized by a sign-uniformity condition. In this work, we use switch operations on signed graphs to characterize unate BN whose signed dependency graph can be transformed into that of a uniform BN. Since switching preserves dynamics up to isomorphism, the dynamics of such BN can be studied through their uniform (simpler) representatives. We apply our results to published BN models of genetic networks, showing that many heterogeneous models can be analyzed through uniform BN.

Keywords: Uniform Boolean networks · Unate Boolean networks · Signed graphs · Switch equivalence · Dynamical isomorphism

A deterministic unate Boolean network (BN) with n entities is a discrete dynamical system defined by a map

$$\mathbf{F} = (F_1, \dots, F_n) : \{0, 1\}^n \rightarrow \{0, 1\}^n,$$

where each local updating function F_i is either increasing or decreasing monotone on each of its variables [3]. In general, the main goal in the study of BN is to understand their dynamics, namely by means of fixed points and limit cycles.

Unate BN can be represented by a signed dependency graph (G, σ) where the sign of an arc, $\sigma(i, j)$, indicates whether the variable x_i has an increasing (+1) or a decreasing (−1) influence on the local function F_j . Uniform BN are a relevant subclass of unate BN defined by a sign-consistency condition [1]. Namely, a BN is said to be uniform if it is either out-uniform or in-uniform: in the first case, all arcs leaving each vertex i have the same sign; in the second case, all arcs entering each vertex i have the same sign. Several well-known families are uniform BN, including homogeneous BN [4,5], or {AND, OR, NAND, NOR}-BN [2].

In this work, we use the *switch operation* on signed graphs [7] to compare BN whose dynamics are preserved up to isomorphism. This provides a way to transfer the dynamical information from well-understood classes to broader and more heterogeneous families of BN. Specifically, we prove that in-uniform and out-uniform BN representatives always appear together within the same switch-equivalence class. Moreover, we characterize the unate BN that are switch-equivalent to a uniform BN, thus making many theoretical results for uniform BN applicable to more heterogeneous classes of unate BN. The same approach can be applied to other sign-defined classes of BN, such as symmetric BN.

Finally, we apply our results to published BN models of gene regulatory networks [6], showing that a significant number of such BN models can be analyzed, after switching, through their uniform representatives.

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